MATH 2050C Lecture 8 (Feb 10)

[Problem Set 4 is posted, due on Feb 18.]

Recall: $\lim(x_n) = x$ iff

$$\forall \varepsilon > 0, \ \exists K \in \mathbb{N} \text{ s.t. } |x_n - x| < \varepsilon \quad \forall n \geq K$$

Q: Given $\varepsilon > 0$, how to find such a K?

small — large

Some useful tools (inequalities)

(i) "Fraction comparison" (everything > 0)

$$\frac{\text{smaller}}{\text{Bigger}} \leq \frac{\square}{\square} \leq \frac{\text{Bigger}}{\text{smaller}}$$

(ii) Triangle ineq. / Reverse Triangle ineq. / AM-GM ineq.

(iii) Bernoulli's ineq.: $(1+x)^n \geq 1 + nx$ $\quad \forall x \geq -1$, $\forall n \in \mathbb{N}$

Example 1 (revisited)

Let $b \in (0,1)$ be fixed. Then

$$\boxed{\lim(b^n) = 0}$$

Pf: Since $b \in (0,1)$, we can write it in the form

$$b = \frac{1}{1+a} \quad \text{for some } a > 0$$

$(\because b < 1)$

Raise to n^{th} power, apply Bernoulli's ineq.

$$b^n = \left(\frac{1}{1+a}\right)^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na}$$

By the example on Tuesday.

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K > \frac{1}{a\varepsilon}$. Then $\forall n \geq K$

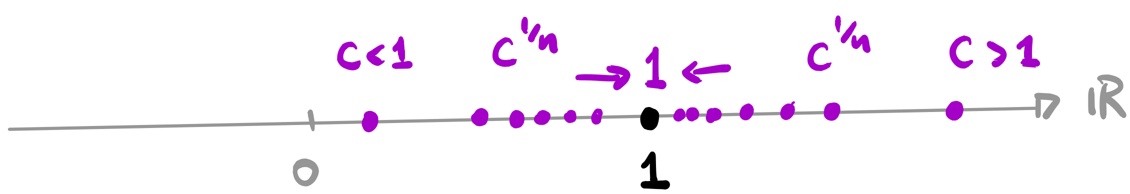
$$|b^n - 0| \leq \frac{1}{1+na} \leq \frac{1}{1+Ka} < \frac{1}{Ka} < \varepsilon$$

$\square$

Example 2: Let $c > 0$ be fixed. Then,

$$\boxed{\lim (c^{\frac{1}{n}}) = 1}$$

Picture:



Proof: Case 1: $c = 1$ then $(c^{\frac{1}{n}}) = (1)$ const seq. ✓

Case 2: $c > 1$ ✓

Recall: $c^{\frac{1}{n}} > 1 \quad \forall n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$,

$$c^{\frac{1}{n}} = 1 + d_n \quad \text{for some } d_n > 0$$

Raise to n^{th} power, apply Bernoulli.

$$c = (c^{\frac{1}{n}})^n = (1 + d_n)^n \overset{\because d_n > 0}{\geq} 1 + n d_n$$

Rearrange. $$d_n \leq \frac{c-1}{n} \quad \text{———} \quad (*)$$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ st. $K > \frac{c-1}{\varepsilon}$.

When $n \geq K$, we have

$$\left| c^{\frac{1}{n}} - 1 \right| = |d_n| = d_n \leq \frac{c-1}{n} \leq \frac{c-1}{K} < \varepsilon$$

Case 3: $0 < c < 1$

Recall: $0 < c^{\frac{1}{n}} < 1 \quad \forall n \in \mathbb{N}$.

Write: $$c^{\frac{1}{n}} = \frac{1}{1 + h_n} \quad \text{where } h_n > 0$$

Raise to n^{th} power.

$$c = (c^{\frac{1}{n}})^n = \frac{1}{(1+h_n)^n} \leq \frac{1}{1+nh_n} < \frac{1}{nh_n}$$

Rearrange. $h_n < \frac{1}{cn}$ for all $n \in \mathbb{N}$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K > \frac{1}{c\varepsilon}$.

For $n \geq K$,

$$|c^{\frac{1}{n}} - 1| = \left| \frac{1}{1+h_n} - 1 \right| = \left| \frac{-h_n}{1+h_n} \right|$$

$$= \frac{h_n}{1+h_n} < h_n < \frac{1}{cn} \leq \frac{1}{cK} < \varepsilon$$

$\square$

<u>Example 3</u>: $\boxed{\lim (n^{\frac{1}{n}}) = 1}$

<u>Pf</u>: Recall: $1 \leq n^{\frac{1}{n}} \quad \forall n \in \mathbb{N}$. Write $n^{\frac{1}{n}} = 1 + \underbrace{\boxed{k_n}}^{\geq 0}$

$$\Rightarrow \quad n = (1+k_n)^n \underset{(\because k_n \geq 0)}{\geq} 1 + \frac{1}{2}n(n-1)k_n^2$$

Rearrange $k_n^2 \leq \frac{n-1}{\frac{1}{2}n(n-1)} = \frac{2}{n}$.

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t $K > \frac{2}{\varepsilon^2}$. Then $\forall n \geq K$,

$$|n^{\frac{1}{n}} - 1| = |k_n| = k_n \leq \sqrt{\frac{2}{n}} \leq \sqrt{\frac{2}{K}} < \varepsilon$$

$\square$